Noncommutative Geometry

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Noncommutative Spaces

It was noticed a long time ago that various properties of sets of points can be restated in terms of properties of certain commutative rings of functions over those sets. In particular, this observation proved to be extremely fruitful in algebraic geometry and has led to tremendous progress in this subject over the past few decades. In these developments the concept of a point in a space is secondary and overshadowed by the algebraic properties of the (sheaves of) rings of functions on those spaces.

This idea also underlies *noncommutative geometry*, a new direction in mathematics initiated by the French mathematician Alain Connes and outlined in his recent book [3]. In noncommutative geometry one goes one step further: it is no longer required that the algebra of functions be commutative! Furthermore, while algebraic geometry did not entirely rid itself of the concept of a point, noncommutative geometry does not use this concept at all. In fact, a point in a noncommutative space is often a contradiction in terms.

One of the sources of noncommutative geometry is the following classic theorem due to Gelfand and Naimark.

Theorem 1. Let $\mathfrak{A}$ be a commutative $\mathbb{C}^*$-algebra, and let M denote the set of maximal ideals of $\mathfrak{A}$. Then, equipped with a natural topology, M is a locally compact space, and $\mathfrak{A} \simeq C_0(M)$, where $C_0(M)$ denotes the $\mathbb{C}^*$-algebra of continuous functions on M vanishing at infinity.

Recall that a $\mathbb{C}^*$-algebra is an algebra over $\mathbb{C}$, equipped with an involutive operation $*$ and a norm $\|\cdot\|$, which satisfies the condition $\|SS^*\| = \|S\|^2$. In other words, the category of locally compact spaces is equivalent to the category of abelian $\mathbb{C}^*$-algebras. The points of a topological space can be characterized in purely algebraic terms as the maximal ideals of an algebra of functions on the space.

Another important source of inspiration for noncommutative geometry is quantum physics. It has been known since the heroic days of quantum mechanics (Heisenberg, Born, Jordan, Schrödinger, Dirac, von Neumann, ...) that ordinary concepts of classical mechanics and symplectic geometry do not apply to the subatomic world. In order to understand the physical phenomena taking place at the atomic scale, one needs to replace the concepts of classical geometry by other, noncommutative structures. The notion of a function on phase space needs to be replaced by an operator acting on a Hilbert space of states $\mathcal{H}$ or a quantum *observable*. In Dirac's parlance, *c-numbers* get replaced by *q-numbers*. This procedure is called *quantization*.

The simplest example is that of a flat space $\mathbb{R}^2$ which is the phase space of a particle moving in one dimension. After quantization, the coordinates q and p of a point in $\mathbb{R}^2$ are replaced by operators q and p which obey the Heisenberg-Born-Jordan commutation relation

$$[q, p] = i\hbar I, \tag{1}$$

where $\hbar$ is a fundamental constant of nature, Planck's constant. Explicitly, one takes the Hilbert space of states to be $\mathcal{H} = L^2(\mathbb{R})$, and $q\psi(x) = x\psi(x)$, $p\psi(x) = -i\hbar\psi'(x)$. This quantization procedure results in a structure which can be thought of as a noncommutative deformation of a classical phase space. Heisenberg's uncertainty principle implies that there is no natural concept of a point on this quantum deformed phase space: all we have is a nonabelian algebra of "functions on the noncommutative plane".

One can also quantize classical phase spaces with more complicated geometry; the simplest of them is the torus $\mathbb{T}^2$. Here the algebra of observables is generated by two unitary operators u and v, such that

$$uv = e^{2\pi i\lambda} vu, \tag{2}$$

where $\lambda = 2\pi\hbar$. One can think of u and v as quantizations of the classical functions $e^{2\pi ip}$ and $e^{2\pi iq}$, respectively. The resulting noncommutative algebra is called the (algebra of functions on the) *quantum torus* or the *irrational rotation algebra*. This algebra also appears naturally in other contexts (periodic structures in magnetic fields, matrix models of string theory). Quantization of more complicated geometries leads to noncommutative structures which cannot be described as easily.

Related noncommutative structures arise as q-deformations of Lie groups (see, e.g., [5, 8]). These structures are often called *quantum groups*.[1] Here the noncommutative algebras carry the additional structure of a *Hopf algebra*. This extra structure on the algebra of functions encodes the fact that the underlying noncommutative space is a group-like object. To illustrate this, let us consider the abelian case first. Let $\mathfrak{F}$ be an algebra (whose topological structure we ignore) of complex functions on a group G with the identity element e. For $f \in \mathfrak{F}$ we set $\Delta f(g_1, g_2) = f(g_1 g_2)$, $\varepsilon f(g) = f(e)$, and $Sf(g) = f(g^{-1})$. This defines algebra homomorphisms $\Delta : \mathfrak{F} \to \mathfrak{F} \otimes \mathfrak{F}$ ("coproduct") and $\varepsilon : \mathfrak{F} \to \mathbb{C}$ ("counit") and an algebra antihomomorphism $S : \mathfrak{F} \to \mathfrak{F}$ ("antipode"). The usual properties defining a group can equivalently be formulated in the language of these homomorphisms without ever referring to the notion of an element of G. Explicitly, we verify easily that $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$, $(\varepsilon \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \varepsilon) \circ \Delta = \mathrm{Id}$, and $m \circ (S \otimes \mathrm{Id}) \circ \Delta = I\varepsilon$. In the last of these identities, m denotes the multiplication on $\mathfrak{F}$, and I is the identity element of $\mathfrak{F}$. A Hopf algebra is an algebra (not necessarily commutative) which is equipped with homomorphisms Δ and ε and an antihomomorphism S, which satisfies the three conditions stated above. For the case of $SU(2)$, the corresponding quantum group (or q-deformation) is constructed as follows. We let a, b denote the complex valued functions on $SU(2)$ assigning to a group element g the corresponding matrix entry in

$$SU(2) \ni g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

The algebra generated by these functions is abelian. The ($\mathbb{C}^*$-algebra of functions on the) quantum group $SU_q(2)$ is defined as a deformation of the algebra of functions on $SU(2)$. We will denote the generators of the deformed algebra by the same symbols, $a, b, \bar{a}, \bar{b}$, and impose the relations

$$\begin{aligned} ab &= qba, & a\bar{b} &= qa\bar{b}, \\ b\bar{b} &= \bar{b}b, & b\bar{a} &= q\bar{a}b, \\ \bar{b}\bar{a} &= q\bar{a}\bar{b}, & a\bar{a} - \bar{a}a &= \left(q^{-1} - q\right) b\bar{b}, \end{aligned}$$

where q is a real parameter such that $|q| < 1$. The abelian case, that of the algebra of continuous functions on $SU(2)$, corresponds to $q = 1$. For the construction of the relevant Hopf algebra structure we refer the reader to the literature.

Voiculescu's *free probability theory* [7] is another example of a noncommutative structure motivated by physics applications. Here the concept of probability space is replaced by a noncommutative structure leading to noncommuting random variables. One of the main results of this theory, Voiculescu's central limit theorem, yields the Wigner semicircle law, which arises in the theory of random matrices. Related fields of *quantum ergodic theory* and *quantum information theory* have recently been the focus of a great deal of attention. They play a pivotal role in the emerging field of quantum computation.

Interesting examples of noncommutative spaces abound, and they are thoroughly discussed in Connes' book. In fact, it turns out that noncommutative geometry also provides a convenient framework for studying "commutative" but highly singular structures. These include fractal sets and products of smooth manifolds by finite sets.

K-Theory and Fredholm Modules

An important part of Connes' program is the notion of a vector bundle over a noncommutative space. The well-known Swan's theorem states that the algebraic K_0 group of the algebra $C(M)$ (defined in terms of stable isomorphism classes

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[1] *The term "quantum groups" is somewhat misleading, as these structures are not quantizations of classical groups in the physical sense described above.*

of projective $C(M)$-modules), where M is compact, coincides with the Grothendieck group $K(M)$ (defined in terms of locally trivial, finite-dimensional complex vector bundles over M) of the underlying manifold. In the framework of noncommutative geometry one can thus regard the elements of the group $K_0(\mathfrak{A})$, where $\mathfrak{A}$ is now a nonabelian algebra of functions on a noncommutative space, as (equivalence classes of) vector bundles over the noncommutative space.

The K-theory of operator algebras was originated by Brown, Douglas, and Fillmore [1] in the mid-seventies—and, incidentally, they gave the first look into the Connes program. Their goal was to reinterpret the classical K-theory in terms of operator algebras, thus extending K-theory to general topological spaces. They classified all $\mathbb{C}^*$-algebra extensions

$$1 \to \mathcal{K} \to \mathfrak{A} \to C(M) \to 1$$

of $\mathfrak{A}$, where $\mathcal{K}$ is an ideal of compact operators and where $C(M)$ is a commutative $\mathbb{C}^*$-algebra. They showed how to construct a group from such extensions. For M compact and finite dimensional, this group is isomorphic to the Steenrod K-homology of M, $K_1(M)$.

One of the fundamental concepts of K-theory is a *Fredholm module* over an algebra $\mathfrak{A}$. An *odd* Fredholm module is a triple $(\mathcal{H}, \pi, F)$ consisting of a Hilbert space $\mathcal{H}$, a $*$-representation π of $\mathfrak{A}$ by bounded linear operators on $\mathcal{H}$, and a self-adjoint unitary operator F such that $[F, \pi(a)]$ is compact for all $a \in \mathfrak{A}$. A Fredholm module is called *even* if, in addition, the Hilbert space $\mathcal{H}$ is $\mathbb{Z}_2$-graded, meaning that $\mathcal{H}$ comes equipped with a self-adjoint unitary operator γ such that $\gamma F + F\gamma = 0$.

Fredholm modules arise naturally in ordinary differential geometry. Let M be a Riemannian manifold, and let $\Lambda_2^p(M)$ denote the space of square integrable p-forms on M. Set $\mathcal{H} = \bigoplus_p \Lambda_2^p(M)$, and let $\gamma = (-1)^p$. With F denoting the phase of the de Rham operator $d : \Lambda_2^p(M) \to \Lambda_2^{p+1}(M)$, $d = F|d|$, the triple $(\mathcal{H}, \pi, F)$ is an even Fredholm module. Similarly, a Dirac operator D on a spin manifold M defines a Fredholm module.

Fredholm modules also arise in various contexts in physics. Historically, the first example came up in Dirac's theory of the electron and, in fact, had a profound impact on modern index theory. More recently, speculations about high-energy physics led to the general concept of *supersymmetry*. Supersymmetry is a deep generalization of the usual space-time symmetries which treats bosons (particles with integer spins, for example, photons) and fermions (particles with half-integer spins, for example, electrons) on equal footing. Mathematically this amounts to replacing the concept of a Lie group by a $\mathbb{Z}_2$-graded concept, that of a *Lie supergroup*. The latter is a noncommutative space whose coordinate algebra contains anticommuting elements. A particular combination of the generators of supersymmetry acting on the Hilbert space of states of the system is an infinite-dimensional, Dirac-type operator.

Other examples of Fredholm modules include models associated with physical phenomena of quantum Hall effect and quantum chaos.

Indeed, much of classical differential geometry can be encoded into the concept of a Fredholm module. The length of a curve on a manifold M is defined in terms of a Riemannian structure on the manifold. If $\gamma : [0, 1] \to M$ is a smooth curve, then its length is $L(\gamma) = \int_0^1 \sqrt{g_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu}\, dt$, where g is a metric tensor. The distance $d(p, q)$ between two points p and q on M is given by $\inf_\gamma L(\gamma)$, where the infimum is taken over all smooth curves γ connecting p and q. For M a compact spin manifold with the Dirac operator D, Connes notices that this distance can be expressed in terms of a Fredholm module over $C(M)$: $d(p, q) = \sup_f |f(p) - f(q)|$, where the supremum is taken over all smooth f such that the commutator $[f, D]$ is bounded and satisfies $\|[f, D]\|_\infty \le 1$. Extrapolating this observation to the noncommutative case, we can thus regard a Fredholm module over an algebra as defining the metric structure on the underlying noncommutative space. This fact underlies, among other things, Connes's formulation of the standard model of elementary interactions.

Cyclic Cohomology and Index Theory

One of the cornerstones of noncommutative geometry is *cyclic cohomology*. This cohomology theory is a far-reaching generalization of the classical de Rham theory. Originally it was developed by Connes [2] and, independently, by Boris Tsygan [6] in the early eighties. Cyclic cohomology is a refinement of Hochschild cohomology and is constructed as follows. We begin by defining the *Hochschild cohomology* of the algebra $\mathfrak{A}$ (technically, what I describe below is the Hochschild cohomology of $\mathfrak{A}$ with coefficients in the dual $\mathfrak{A}^*$).

Let $C^n(\mathfrak{A}, \mathfrak{A}^*)$ be the space of complex-valued, $(n+1)$-linear functionals on $\mathfrak{A}$, whose elements we will denote by f. We consider the operator $b : C^n(\mathfrak{A}, \mathfrak{A}^*) \to C^{n+1}(\mathfrak{A}, \mathfrak{A}^*)$ defined by

$$\begin{aligned}(bf)&(a_0, a_1, \ldots, a_{n+1}) \\ &= \sum_{j=0}^{n} (-1)^j f(a_0, \ldots, a_j a_{j+1}, \ldots a_{n+1}) \\ &+ (-1)^{n+1} f(a_{n+1} a_0, a_1, \ldots, a_n).\end{aligned}$$

One verifies that $b \circ b = 0$, so that b is a coboundary operator. The cohomology of the complex $(C^*(\mathfrak{A}, \mathfrak{A}^*), b)$ is called the Hochschild cohomology of $\mathfrak{A}$ and is denoted by $H^*(\mathfrak{A}, \mathfrak{A}^*)$. In the case of $\mathfrak{A} = C^\infty(M)$, where M is a smooth manifold, the Hochschild cohomology groups reproduce the spaces of de Rham currents on M. That is in a way a drawback, since one would like a cohomology scheme which reproduces, in the commutative case, the de Rham homology.

To construct such a theory, we let $C_\lambda^n(\mathfrak{A})$ denote the space of those cochains $f \in C^n(\mathfrak{A}, \mathfrak{A}^*)$ for which the following cyclicity condition holds:

$$f(a_n, a_0, \ldots, a_{n-1}) = (-1)^n f(a_0, a_1, \ldots, a_n).$$

Then $(C_\lambda^*(\mathfrak{A}), b)$ turns out to be a subcomplex of $(C^*(\mathfrak{A}, \mathfrak{A}^*), b)$, and its cohomology is called the cyclic cohomology of $\mathfrak{A}$. The cyclic cohomology groups are denoted by $HC^n(\mathfrak{A})$. The theorem below shows that cyclic cohomology has the desired property.

Theorem 2. Let M be a smooth compact manifold, and let $\mathfrak{A} = C^\infty(M)$ denote the algebra of smooth functions on M. Then there is a canonical isomorphism

$$HC^n(\mathfrak{A}) \simeq \ker(b) \oplus \bigoplus_j H_{n-2j}(M),$$

where $H_k(M)$ denotes the de Rham homology of M.

The cyclic cohomologies of a variety of other noncommutative spaces (quantum tori, quantum groups, ...) have been computed.

Suppose now that $(\mathcal{H}, \pi, F)$ is an even Fredholm module over $\mathfrak{A}$, with $\mathbb{Z}_2$-grading on $\mathcal{H}$ defined by γ. Associated with this Fredholm module, is a fundamental cyclic cohomology class $\mathrm{ch}(F)$, called the *Chern character*. It is given by

$$\begin{aligned}&\mathrm{ch}(F)(a_0, \ldots, a_n) \\ &= \mathrm{tr}(\gamma a_0 [F, a_1] \ldots [F, a_n]).\end{aligned} \tag{3}$$

This general concept of the Chern character leads to an index theorem which is a profound extension of the classical Atiyah-Singer index theorem. The topological part of the index formula is given in terms of a pairing between $\mathrm{ch}(F)$ and a $K_0(\mathfrak{A})$ class.

Quantum Field Theory

Before describing the content of the last chapter of Connes' book, devoted largely to his formulation of the standard model of fundamental interactions, I would like to present some issues currently faced by relativistic quantum field theory.

Historically, quantum field theories arose as results of "second quantization" of classical systems. The term "first quantization" (or simply, quantization) refers to a procedure which leads from a classical mechanical description of a system to a quantum mechanical description. The latter is usually formulated in terms of a fundamental partial differential equation ("the wave equation") involving a (scalar, vector, spinor, ...) field over the space-time and thus may be thought of as classical field theory. Planck's constant $\hbar$ enters the wave equation, and the classical theory is recovered as a suitable limit as $\hbar \to 0$. The most famous example of this procedure is Schrödinger's quantization of Newtonian mechanics, with the wave equation bearing his name. The first quantization of electromagnetism was carried through by Maxwell in the nineteenth century. His theory marked a transition from corpuscular to wave description of light (the absence of Planck's constant in that theory is a fluke due entirely to the masslessness of the photon). Unlike Schrödinger's theory, Maxwell's theory is relativistic, meaning that it is compatible with the special theory of relativity. Another famous example of a classical field theory is Dirac's quantization of the electron.

Classical field theories do not allow one to describe systems in which the number of particles is not fixed (particles get created or annihilated as a result of interaction), and thus they fail to apply to high energy physics. A procedure leading to a description of systems with a variable number of particles is called "second quantization" because of its similarity to quantization. The starting point of second quantization is a first quantized (classical) field theory, and the result is a quantized field theory. If the classical field theory is relativistic, the resulting quantum field theory is called relativistic quantum field theory. The table below summarizes these remarks.

$$\begin{array}{c} \text{Particles} \\ \begin{array}{ccc} \text{Classical mechanics} & & \text{Geometric optics} \\ \downarrow & & \downarrow \\ & \text{Classical fields} & \\ \text{Quantum mechanics} & & \text{Maxwell's theory} \\ \downarrow & & \downarrow \\ & \text{Quantum fields} & \\ \text{Second quantized theory} & & \text{Quantum electrodynamics} \end{array} \end{array} \tag{4}$$

Shortly after relativistic quantum field theory was discovered, it became clear that it suffered from serious conceptual and technical deficiencies. The only known way of extracting any information from the fundamental equations was to expand in a power series in the coupling constant. Such an expansion is called *perturbation theory*. Unfortunately, it turns out that each term of perturbation theory, except for the leading order contribution, is singular, and so the power series as it stands is meaningless! The path-breaking work of Feynman, Schwinger, Dyson, and others resulted in a procedure of extracting finite parts of the singular expressions encountered in perturbation theory known as *renormalization theory*. Renormalization theory removes singularities from perturbation theory at the expense of introducing a number of arbitrary constants whose values should be determined by experiment. That leads one to the requirement that the number of such constants should be finite (otherwise one could "explain" any experiment) and that they should be measurable parameters of the theory. Any quantum field theory satisfying these requirements is called *renormalizable*. These concepts led to some of the most remarkable developments in physics. The theory of interacting electrons and photons, quantum electrodynamics, turns out to be renormalizable and leads to fantastic agreement with experiment.

The requirement of renormalizability became a paradigm of quantum field theory, and it proved extremely fruitful. Guided by it, particle physicists generalized quantum electrodynamics to include other types of interactions: the Weinberg-Salam model unifying electromagnetic and weak interactions and the standard model unifying electromagnetic, weak, and strong interactions. As of today there is a consensus that the standard model is the correct theory of elementary interactions. One disturbing fact about this theory is that gravity has so far resisted inclusion into the framework of renormalizable quantum field theory.

This renormalizability paradigm underwent a dramatic revision in the seventies as a result of Wilson's renormalization group theory. According to Wilson, we do not need to know the details of the "true" theory of elementary interactions which is valid at all energy scales. It may as well be that the fundamental theory is not a quantum field theory at all. All we have at our disposal is an effective theory, a low energy limit of this fundamental theory. The concept of renormalizability thus acquires a new meaning: renormalizable theories, rather than being "fundamental", are merely those theories which survive the scaling down from the fundamental scale to the "laboratory" scale. Nonrenormalizable theories get wiped out in the process of taking this limit.

The Standard Model à la Connes

The last chapter of Connes' book contains an account of his formulation of the standard model of fundamental interactions. Anybody who has studied the standard model in its usual formulation with all of its plethora of fields and mechanisms will appreciate the compact geometric formulation presented in the book. Connes' formulation provides a natural geometric principle underlying this theory within the framework of noncommutative geometry.

The first basic idea of Connes' theory, in its simplest version (which does not yield the full standard model), consists of the following. Consider a two-sheeted space time $\mathbb{R}^4 \times \mathbb{Z}_2$, where the $\mathbb{R}^4$ factor is the usual space time and the $\mathbb{Z}_2$ factor is a "discrete dimension" (this resembles slightly the Kaluza-Klein theory unifying Einsteinian gravity and electromagnetism). The algebra of smooth functions on this space, $C^\infty(\mathbb{R}^4) \oplus C^\infty(\mathbb{R}^4)$, is abelian. However, the derivative in the discrete direction is a finite difference quotient, and so a Dirac operator D on this space is somewhat unusual. More refined versions of the theory (to reproduce the correct quark content of the standard model) involve multiplying $\mathbb{R}^4$ by more complicated finite sets and setting up a sophisticated bimodule structure over the algebra of functions on the resulting product space.

This is where the formalism of noncommutative geometry becomes crucial. Using his algebraic framework, Connes develops the relevant gauge theory associated with (the Fredholm module defined by) D. This leads to a natural concept of the connection form $\mathcal{A}$, curvature form $\mathcal{F}$, etc.

The second basic idea in Connes' approach is the use of the *Dixmier trace* as the fundamental functional to define the action of the theory. The Dixmier trace of a positive operator S with discrete spectrum λ_j is given by

$$\mathrm{tr}_\omega(S) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{1\le j\le N} \lambda_j$$

(so it is nontrivial only for operators with logarithmically divergent traces). The Yang-Mills action functional of the theory is now given by $\mathrm{tr}_\omega(|D|^{-4}\mathcal{F}^2)$. The remarkable fact is that this action functional plus the fermionic action functional which I am not discussing here reproduce the action functional of the standard model. To acknowledge the depth of this fact, one needs to go through a truly excruciating series of algebraic manipulations.

Connes' theory is purely classical, and so it belongs to the middle column of the table (4). He does not propose a quantization scheme which would be intrinsic to it; the only known way to obtain a quantum field theory out of Connes' model is to follow the usual rules of quantum field theory. Because of its classical character, the model (or its more recent refinements, which include Einsteinian gravity [4]) does not resolve the issues described in the previous section, and it does not provide any new physical principle toward this goal. No unified theory of fundamental interactions is known; the consensus among physicists is that string theory is currently the only promising attempt at such a theory. Despite this, the Connes formulation of the standard model is a truly remarkable construction and can be regarded as a major triumph of noncommutative geometry.

Let me mention that some recent developments in string theory show some striking and mysterious similarities between Connes' theory and the theory of D-branes. Is it possible that noncommutative geometry underlies the fundamentals of string theory?

Final Remarks

It is an impossible task for me to discuss in a short review the wealth of fascinating mathematical phenomena described by Alain Connes. I focused on the aspects of noncommutative geometry which I understand best, and these are its relationships with theoretical physics.

Connes' *Noncommutative geometry* is one of the milestones of mathematics. It lays the foundations of a new branch of mathematics whose importance is difficult to overestimate. Its impact will be felt by generations of mathematicians to come, the way Riemann's *Über die Hypothesen* ... influenced the development of differential geometry.

The book has a largely programmatic, expository character: few things are proved, but the presentation is extremely lucid. It is a source of ideas, inspiration, ingenious calculations, and facts for researchers who are interested in one of the most fascinating developments in mathematics.

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